The Hindman theorem for semigroups

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Introduction

Designations

- $\forall n \in N\colon \bar{n} = \{1, \ldots, n\}$
- $\forall X\colon 2^X_\circ = \{A \in 2^X \mid A \text{ is finite}\}$
- $\forall X\colon 2^X_\infty = \{A \in 2^X \mid A \text{ is infinite}\}$
- *for any semigroup* $(S, \circ)$ *and each* $a, s, t \in S$ $rW_s^{t,a} = \{n \in N \mid s \circ t^n = a\}$
- *for any semigroup* $(S, \circ)$ *and each* $a, s, t \in S$ $lW_s^{t,a} = \{n \in N \mid t^n \circ s = a\}$

Introduction

Designations

- $\beta X = \{\emptyset \neq F \subset 2^X \mid F \text{ is ultrafilter}\} \subset 2^{2^X}$
- $\beta X_\circ^{id} = \{F \in \beta X \mid F \circ F = F\} \subset 2^{2^X}$ *for any semigroup* $(X, \circ)$
- $\beta^g X = \{\emptyset \neq F \subset 2^X \mid F \text{ is principal ultrafilter}\} \subset 2^{2^X}$
- $\beta^w X = \{\emptyset \neq F \subset 2^X \mid F \text{ is free ultrafilter}\} \subset 2^{2^X}$
- $^\star \colon 2^X \to 2^{\beta X}; A \mapsto A^\star = \{F \in \beta X \mid A \in F\} \subset \beta X$
- $e \colon X \to 2^{2^X}; a \mapsto e(a) = \{A \in 2^X \mid a \in A\} \subset 2^X$

Introduction to Hindman theorem

We would like to discuss the proof of Hindman theorem that states as follows.

Theorem

For any division of N into finite number of trays we always find such tray that there is an infinite $A \subset N$ that is in this tray and all finite sums of elements of the set A belongs to this tray.

There is written a lot of proofs for this theorem but we are going to concentrate on the proof that uses the Cech Stone compactification of N. In fact this is the shortest proof.

It turns that the same construction as in Cech Stone compactification of N works for general infinite semigroups $(S, \circ)$. These semigroups have an additional property called movability. There is proven that for an infinite rightmoving semigroup the Hindman theorem holds since this is equivalent to the fact that the Cech Stone remainder $\beta S - S$ is subsemigroup of βS. But what is happen if this semigroup if leftmoving?

Moving semigroups

Definition

The infinite semigroup $(S, \circ)$ is rightmoving$\Leftrightarrow \forall F \in 2^S_\circ : \forall A \in 2^S_\infty : \exists B \in 2^A_\circ : \{s \in S \mid B \circ s \subset F\} \in 2^S_\circ$

Definition

The infinite semigroup $(S, \circ)$ is leftmoving$\Leftrightarrow \forall F \in 2^S_\circ : \forall A \in 2^S_\infty : \exists B \in 2^A_\circ : \{s \in S \mid s \circ B \subset F\} \in 2^S_\circ$

There are The Eight Immortal Theorems about moving semigroups that we are introducing here.

What semigroups are not moving

Theorem

For each infinite semigroup $(S, \circ)$ *and for each such* $t, a \in S$ *that* $\forall m, n \in N\colon m \neq n \Rightarrow t^n \neq t^m$ *and* $t \neq a$ *holds:*

- *if* $\forall s \in S - \{t\}\colon rW_s^{t,a} \in 2^N_\infty$ *and* $(rW_s^{t,a})_{s \in S - \{t\}}$ *has finite infinite intersection property then* $(S, \circ)$ *is not rightmoving.*
- *if* $\forall s \in S - \{t\}\colon lW_s^{t,a} \in 2^N_\infty$ *and* $(lW_s^{t,a})_{s \in S - \{t\}}$ *has finite infinite intersection property then* $(S, \circ)$ *is not leftmoving.*

Jump to proof

What semigroups are rightmoving and leftmoving?

Theorem

- *Every infinite semigroup* $(S, \circ)$ *such that* $\forall a \in S\colon L_a$ *is finite to one is rightmoving.*
- *Every infinite semigroup* $(S, \circ)$ *such that* $\forall a \in S\colon R_a$ *is finite to one is leftmoving.*
- *Every infinite leftcancellative semigroup* $(S, \circ)$ *is rightmoving.*
- *Every infinite rightcancellative semigroup* $(S, \circ)$ *is leftmoving.*
- *Every infinite rightcancellative semigroup* $(S, \circ)$ *is rightmoving.*
- *Every infinite leftcancellative semigroup* $(S, \circ)$ *is leftmoving.*

Jump to proof

What is Cech Stone compactification

Definition

For every topological Hausdorff space (X, τ) the compactification is the pair $(\phi, C) \Leftrightarrow C$ is compact$\wedge \phi \colon X \to C$ is homeomorphism$\wedge \phi(X)$ is dense in C.

Among all compactifications the Cech Stone compactification is the richest and has the highest power.

Definition

For every topological regular space (X, τ_X) and its compactification (ϕ, Z) the compactification (ϕ, Z) is Cech Stone$\Leftrightarrow \forall (Y, \tau_Y)$ compact topological space: $\forall f \in C(X, Y) \colon \exists g \in C(Z \supset \phi(X), Y) \colon g \circ \phi = f$.

The Cech Stone compactification is unique

Theorem

For every topological space (X, τ) and every two different Cech Stone compactification of this space (ϕ, W) and (ψ, V) there is such $g \in C(V \supset \psi(X), \phi(X) \subset W)$ that $g \circ \psi = \phi$.

Therefore every Cech Stone compactification of the same space (X, τ) can be brought to one and for countable space X we can deal only with Cech Stone compactification that is made with ultrafilters. Now assume that we know without proofs that $(e, \beta N)$ is the Cech Stone compactification of N. Jump to proof

The semigroup structure

βN becomes a continuous semigroup under the special addition.

Definition

$+ \colon \beta N^2 \to 2^{2^N}; (F, G) \mapsto F + G = \{A \subset N \mid \{n \in N \mid A - n \in G\} \in F\}.$

Theorem

- $+ \colon \beta^g N^2 \to \beta^g N; (F, G) \mapsto F + G = \{A \subset N \mid \{n \in N \mid A - n \in G\} \in F\}$
- $\forall F, G \in \beta N \colon F + G \in \beta N$
- $\forall F, G, H \in \beta N \colon (F + G) + H = F + (G + H)$
- $\forall G \in \beta N \colon +_G \colon \beta N \to \beta N; F \mapsto +_G(F) = F + G$ *is continuous function.*

Jump to proof

Idempotent ultrafilters

The focal point of the Cech Stone compactification of any discrete space X which is used in the proof of Hindman theorem is idempotence. The following equalities from ultrafilters permit the use of $G-$pair in proof of Hindmann theorem.

Theorem

- $\beta^{id}(N) \neq \emptyset$
- $\beta^{id}(N) \cap \beta^{g}(N) = \emptyset$
- $\forall F \in \beta^{id} N \colon \forall A \in 2^{N} \colon A \in F \Leftrightarrow \hat{A} \in F$
- $\forall F \in \beta^{id} N \colon \forall A \in F \colon A \cap \hat{A} \in F$
- $\forall F \in \beta^{id} N \colon \forall A \in F \colon A \cap \hat{A} \neq \emptyset$
- $\forall F \in \beta^{id} N \colon \forall A \in F \colon \exists a \in X \colon A \cap (A - a) \in F$
- $\forall F \in \beta^{id} N \colon \forall A \in F \colon \exists a \in X \colon A \cap (A - a) - \{a\} \in F$

Jump to proof

$G-$pair

Now we define the special tool for proving Hindmann theorem.

Definition

For every $F \in \beta^{id} N$ and every $A \in F$ the pair of sequences $(K_n) \colon N_0 \to F; n \mapsto K_n$ and $(a_n) \colon N_0 \to N; n \mapsto a_n$ is $G-$pair starting from A if and only if

- $K_n = \begin{cases} A \Leftrightarrow n = 0 \\ K_{n-1} \cap (K_{n-1} - a_n) - \{a_n\} \Leftrightarrow n \in N \end{cases}$
- $a_n = \begin{cases} 1 \Leftrightarrow n = 0 \\ \min(K_{n-1} \cap \hat{K_{n-1}}) \Leftrightarrow n \in N \end{cases}$

Here we express a certain property that solves Hindman theorem. To prove the Hindman theorem we need only the starting set.

Theorem

- *For every $F \in \beta^{id}N$ and every $A \in F$ the $G-$pair (K_n, a_n) starting from A fulfills $\forall n \in N: K_n \subset K_{n-1} \wedge a_n + K_n \subset K_{n-1}$.*

Jump to proof

Here we recall the Hindmann theorem.

Theorem

For any division of N into finite number of trays we always find such tray that there is an infinite $A \subset N$ that is in this tray and all finite sums of elements of the set A belongs to this tray.

It is enough to indicate the starting set for $G-$pair. Let A will be one of trays.

Proof of Hindman theorem

Proof.

Recall that $\beta^{id}N \neq \emptyset$. Therefore there is such $F \in \beta N$ that $F = F + F$. Now we name the starting set for $G-$pair. Now divide the set N on a finite number $n \in N$ of trays. In this way we get a finite collection of disjoint sets $(A_1, \ldots, A_n)$. These sets are the division of set N. Because there is $F \ni N = \bigcup_{i \in \bar{n}} A_i$ and F is a prime filter so there is only one such $i \in \bar{n}$ that $A_i \in F$. Let (K_n, a_n) is $G-$pair starting from $A_i \in F$. For example check that $a_{11} + a_5 + a_3$ is in the same tray as $a_{11}, a_5, a_3, a_{11} + a_5, a_3 + a_{11}, a_5 + a_3$. Notice that $a_{11} \in K_{10} \subset K_4$ i $a_5 \in K_4$ and therefore $a_{11} + a_5 \in K_4 \subset K_2$ because $a_5 + K_5 \in K_4$ and $a_3 \in K_2$ and therefore $a_{11} + a_5 + a_3 \in K_2 \subset K_0$ that means $a_{11} + a_5 + a_3$ are in one tray. Our infinite set from the theorem of Hindmann is $\{a_n \mid n \in N\}$. $\square$

The mixing hypothesis asks whether it can be so that the following theorem holds.

Theorem

For any division of N into finite number of trays we always find such tray that there is an infinite $A \subset N$ that is in this tray and all finite sums of elements of the set A belongs to this tray and moreover all finite multiplications of elements of the set A belongs to this tray.

Mixing hypothesis

In the paper 'Ultrafilters with applications to analysis, social choice and combinatorics' of Galvin is written that the following problem is not solved to this day.

Problem

For every $n \in N$ if we divide the set N on n numbered trays from the set $\bar{n}$ then there is such number $k \in \bar{n}$ and such numbers $x, y \in N$ that $x, y, x + y, x \cdot y$ are in tray number k.

This is because it was shown in the paper 'Sums equal to products in βN' of Hindmann that $\forall F \in \beta N\colon F + F \neq F \cdot F$.

Mixing hypothesis

Remark

Notice that $\forall F \in \beta N\colon F + F \neq F \cdot F \Rightarrow \beta^{id}_{+} N \cap \beta^{id}_{\cdot} N = \emptyset$. *To see this assume that*
$\exists F \in \beta N\colon F \in \beta^{id}_{+} N \cap \beta^{id}_{\cdot} N \Leftrightarrow \exists F \in \beta N\colon F + F = F \wedge F = F \cdot F.$
Therefore $\exists F \in \beta N\colon F + F = F \cdot F$.

Mixing hypothesis

We are going to solve mixing hypothesis with false assumption that there is such $F \in \beta N$ that $F \in \beta_{+}^{id} N \cap \beta_{\cdot}^{id} N$. The procedure is very similar to the already shown addition in βN. As in the case of $A - x = \{y \in X \mid x + y \in A\}$ i $\hat{A}^{+} = \{n \in N \mid A - n \in F\}$ we define $A \cdot \frac{1}{x} = \{y \in X \mid y \cdot x \in A\}$ and $\hat{A}^{\cdot} = \{n \in N \mid A \cdot \frac{1}{n} \in F\}$. As in the case of $+ \colon \beta N^2 \to \beta N; (F, G) \mapsto F + G = \{A \subset N \mid \hat{A}^{+} \in G\} \in F\}$ is well defined and fulfills all theorems of the above we show that $\cdot \colon \beta N^2 \to \beta N; (F, G) \mapsto F \cdot G = \{A \subset N \mid \hat{A}^{\cdot} \in G\} \in F\}$ is well defined and fulfills all mentioned above theorems with small change $+$ to $\cdot$.

Mixing hypothesis

Now we assume that there is such $F \in \beta N$ that $F \in \delta N := \beta_{+}^{id} N \cap \beta_{\cdot}^{id} N$. Notice that for every $F \in \delta N$ holds $\forall A \subset X \colon A \in F = F + F \Leftrightarrow \hat{A}^{+} \in F \wedge A \in F = F \cdot F \Leftrightarrow \hat{A}^{\cdot} \in F$. Therefore for every $A \in F$ holds $\hat{A}^{+} \in F$ together with $\hat{A}^{\cdot} \in F$. Therefore $A \cap \hat{A}^{\cdot} \cap \hat{A}^{+} \in F$ and therefore $A \cap \hat{A}^{\cdot} \cap \hat{A}^{+} \neq \emptyset$. Therefore there is such $a \in X$ that $a \in A \wedge A - a \in F \wedge A \cdot \frac{1}{a} \in F \wedge A \in F$. Therefore there is such $a \in X$ that $(A - a) \cap (A \cdot \frac{1}{a}) \cap A \in F$ and $(A - a) \cap (A \cdot \frac{1}{a}) \cap A - \{a\} \in F$.

Mixing hypothesis

We define $G-$pair that starts from A as in the case of addition.

Definition

- $K_n = \begin{cases} A \Leftrightarrow n = 0 \\ K_{n-1} \cap (K_{n-1} - a_n) \cap (K_{n-1} \cdot \frac{1}{a_n}) - \{a_n\} \Leftrightarrow n \in N \end{cases}$
- $a_n = \begin{cases} 1 \Leftrightarrow n = 0 \\ \min(K_{n-1} \cap \hat{K}_{n-1}^{+} \cap \hat{K}_{n-1}^{\cdot}) \Leftrightarrow n \in N \end{cases}$.

Mixing hypothesis

As in the case of addition we prove that the following theorem holds.

Theorem

For every $F \in \delta N$ and every $A \in F$ the $G-$pair (K_n, a_n) starting from A fulfills
$\forall n \in N \colon K_n \subset K_{n-1} \wedge a_n + K_n \subset K_{n-1} \wedge a_n \cdot K_n \subset K_{n-1}.$

In this way we have an approach to solve mixing hypothesis.

Proof.

We name the starting set for $G-$pair. Now divide the set N on a finite number $n \in N$ of trays. In this way we get a finite collection of disjoint sets $(A_1, \ldots, A_n)$. These sets are the division of set N. Because there is $F \ni N = \bigcup_{i \in \bar{n}} A_i$ and F is a prime filter so there is only one such $i \in \bar{n}$ that $A_i \in F$. Let (K_n, a_n) is $G-$pair starting from $K_0 = A_i \in F$. Notice that $a_{11} \in K_{10} \subset K_4$ i $a_5 \in K_4$ and because $a_5 + K_5 \subset K_4$ we get that $a_5 + a_{11} \in K_4 \subset K_0 = A_i$ and because $a_5 \cdot K_5 \subset K_4$ we get that $a_5 \cdot a_{11} \in K_4 \subset K_0 = A_i$. Therefore there is such tray A_i that $a_{11}, a_5, a_{11} + a_5, a_{11} \cdot a_5 \in A_i$. □

Mixing hypothesis

This fact $\forall F \in \beta N \colon F + F \neq F \cdot F$ is due to complexity of additive and multiplicative structure of N. But there are a lot of operations in natural numbers. It might be able to fins such operation $(N, \circ)$ that is associative, cancellable and $\beta^{id}_{+} N \cap \beta^{id}_{\circ} N \neq \emptyset$. Then we show that $a, b, a + b, a \circ b$ are in one tray. Hence we should look for these two operations $(N, \circ_1)$ and $(N, \circ_2)$ such that N is semigroup and $\beta^{id}_{\circ_1}(N) \cap \beta^{id}_{\circ_2}(N) \neq \emptyset$. Those operations must be such that Hindmann theorem holds for the semigroups $(N, \circ_1)$ and $(N, \circ_2)$. Therefore both $\circ_1$ and $\circ_2$ must be rightmoving.

Rightmoving semigroups

The problem of Hindmann theorem is solved for semigroups $(S, \circ)$ that are rightmoving.

Theorem

For any rightmoving semigroup $(S, \circ)$ the Hindmann theorem with operation $\circ$ holds.

We dont know what is happen if the semigroup $(S, \circ)$ is leftmoving. There are semigroups that are leftmoving but not rightmoving and for rightmoving semigroups we have the following equivalence.

Theorem

For every semigroup $(S, \circ)$ we get that $(S, \circ)$ is rightmoving$\Leftrightarrow$ the Cech Stone remainder $\beta S - S$ is a subsemigroup of βS.

References

Ultrafilters with application to analysis, social choice and combinatorics, David Galvin, September 2009

Combinatorial Number Theory, Boaz Tsaban, 2014

Sums equal to products in βN, Neil Hindmann, 1980

Algebra in the Cech Stone Compactification, Theory and Application, Neil Hindman, Dona Strauss, 2011

Proof.

Choose $F = \{a\} \in 2^S_\circ$ and $A = S - \{t\} \in 2^S_\infty$. Then taking any $s \in A$ we get that $rW_s^{t,a} = \{n \in N \mid s \circ t^n = a\} \in 2^N_\infty$ and therefore $2^S_\infty \ni \{t^n \mid n \in rW_s^{t,a}\} = \{t^n \in S \mid s \circ t^n = a\} \subset \{u \in S \mid s \circ u = a\} \in 2^S$ and therefore $\{u \in S \mid s \circ u = a\} \in 2^S_\infty$. Now we show that for any finite $B \subset A$ holds $\{u \in S \mid B \circ u \subset F\} \in 2^S_\infty$ that contradicts rightmovability. Taking any $n \in N$ and any $s_1, \dots, s_n \in A$ that are different and denoting $B = \{s_1, \dots, s_n\}$ we get that $\forall i \in \bar{n}\colon rW_{s_i}^{t,a} \in 2^N_\infty$. From the above argument we get that

$\forall i \in \bar{n}\colon 2^S_\infty \ni \{t^n \mid n \in rW_{s_i}^{t,a}\} \subset \{u \in S \mid s_i \circ u = a\} \subset S$. □

Jump to proof

The rest of the proof for rightmoving semigroups

Proof.

Now we intersect over $i \in \bar{n}$ these sets $\{t^n \mid n \in \bigcap_{i\in\bar{n}} rW^{t,a}_{s_i}\} = \bigcap_{i\in\bar{n}}\{t^n \mid n \in rW^{t,a}_{s_i}\} \subset \bigcap_{i\in\bar{n}}\{u \in S \mid s_i \circ u = a\} = \{u \in S \mid \forall i \in \bar{n}\colon s_i \circ u = a\} = \{u \in S \mid s_1 \circ u = a \wedge \ldots \wedge s_n \circ u = a\} = \{u \in S \mid \{s_1, \ldots, s_n\} \circ u = a\} = \{s \in S \mid B \circ s \subset F\} \subset S$. Just assuming that $\bigcap_{i\in\bar{n}}\{t^n \mid n \in rW^{t,a}_{s_i}\} \in 2^S_\infty$ we conclude that $\{s \in S \mid B \circ s \subset F\} \in 2^S_\infty$. The assumption that the $(rW^{t,a}_s)_{s\in S-\{t\}}$ has finite infinite intersection property is sufficient. □

Jump to proof

Proof for leftmoving semigroups

Proof.

Choose $F = \{a\} \in 2^S_\circ$ and $A = S - \{t\} \in 2^S_\infty$. Then taking any $s \in A$ we get that $IW_s^{t,a} = \{n \in N \mid t^n \circ s = a\} \in 2^N_\infty$ and therefore $2^S_\infty \ni \{t^n \mid n \in IW_s^{t,a}\} = \{t^n \in S \mid t^n \circ s = a\} \subset \{u \in S \mid u \circ s = a\} \in 2^S$ and therefore $\{u \in S \mid u \circ s = a\} \in 2^S_\infty$. Now we show that for any finite $B \subset A$ holds $\{u \in S \mid u \circ B \subset F\} \in 2^S_\infty$ that contradicts leftmovability. Taking any $n \in N$ and any $s_1, \ldots, s_n \in A$ that are different and denoting $B = \{s_1, \ldots, s_n\}$ we get that $\forall i \in \bar{n}\colon IW_{s_i}^{t,a} \in 2^N_\infty$. From the above argument we get that $\forall i \in \bar{n}\colon 2^S_\infty \ni \{t^n \mid n \in IW_{s_i}^{t,a}\} \subset \{u \in S \mid u \circ s_i = a\} \subset S$. □

Jump to proof

The rest of the proof for leftmoving semigroups

Proof.

Now we intersect over $i \in \bar{n}$ these sets $\{t^n \mid n \in \bigcap_{i\in\bar{n}} IW^{t,a}_{s_i}\} = \bigcap_{i\in\bar{n}}\{t^n \mid n \in IW^{t,a}_{s_i}\} \subset \bigcap_{i\in\bar{n}}\{u \in S \mid u \circ s_i = a\} = \{u \in S \mid \forall i \in \bar{n}\colon u \circ s_i = a\} = \{u \in S \mid u \circ s_1 = a \wedge \ldots \wedge u \circ s_n = a\} = \{u \in S \mid u \circ \{s_1, \ldots, s_n\} = a\} = \{s \in S \mid s \circ B \subset F\} \subset S$. Just assuming that $\bigcap_{i\in\bar{n}}\{t^n \mid n \in IW^{t,a}_{s_i}\} \in 2^S_\infty$ we conclude that $\{s \in S \mid s \circ B \subset F\} \in 2^S_\infty$. The assumption that the $(IW^{t,a}_s)_{s\in S-\{t\}}$ has finite infinite intersection property is sufficient. □

Jump to theorem

Proof of rightmovability if L_a is finite to one

Proof.

The sentence $\forall a \in S\colon L_a$ is finite to one means that $\forall a, t \in S\colon L_a^{-1}(\{t\}) = \{s \in S \mid L_a(s) = a \circ s = t\}$ is finite. Now assume that $(S, \circ)$ is not rightmoving. This means that there are the finite set $F = \{f_1, \ldots, f_m\} \subset S$ and the infinite set $A \subset S$ such that for every finite $B = \{b_1, \ldots, b_n\} \subset A$ the set $S \supset \{s \in S \mid B \circ s \subset F\}$ is infinite. Notice that for this finite B the following equality $\{s \in S \mid B \circ s \subset F\} = \{s \in S \mid \forall i \in \bar{n}\colon b_i \circ s \in F\} = \bigcap_{i \in \bar{n}} \{s \in S \mid b_i \circ s \in F\} = \bigcap_{i \in \bar{n}} \{s \in S \mid \exists j \in \bar{m}\colon b_i \circ s = f_j\} = \bigcap_{i \in \bar{n}} \bigcup_{j \in \bar{m}} \{s \in S \mid b_i \circ s = f_j\}$ holds. Now notice that $\forall a \in S\colon L_a$ is finite to one and therefore $\forall i \in \bar{n}\colon \forall j \in \bar{m}\colon \{s \in S\colon b_i \circ s = f_j\}$ is finite and sums and intersections of finite set is finite. Therefore $\{s \in S \mid B \circ s \subset F\}$ is finite, contradiction. □

Jump to proof

Proof that leftcancellative semigroup is rightmoving

Proof.

The semigroup $(S, \circ)$ jest leftcancellative$\Leftrightarrow \forall a, b, s \in S \colon a \neq b \Rightarrow L_s(a) = s \circ a \neq s \circ b = L_s(b) \Leftrightarrow \forall s \in S \colon L_s$ is injection. Therefore for every $s \in S$ preimage of singleton $L_s^{-1}(\{t\})$ is at most singleton and therefore $\forall a \in S \colon L_a$ is finite to one and therefore $(S, \circ)$ is rightmoving.

□

Jump to proof

Proof.

The sentence $\forall a \in S\colon R_a$ is finite to one means that $\forall a, t \in S\colon R_a^{-1}(\{t\}) = \{s \in S \mid R_a(s) = s \circ a = t\}$ is finite. Now assume that $(S, \circ)$ is not leftmoving. This means that there are the finite set $F = \{f_1, \ldots, f_m\} \subset S$ and the infinite set $A \subset S$ such that for every finite $B = \{b_1, \ldots, b_n\} \subset A$ the set $S \supset \{s \in S \mid s \circ B \subset F\}$ is infinite. Notice that for this finite B the following equality $\{s \in S \mid s \circ B \subset F\} = \{s \in S \mid \forall i \in \bar{n}\colon s \circ b_i \in F\} = \bigcap_{i \in \bar{n}} \{s \in S \mid s \circ b_i \in F\} = \bigcap_{i \in \bar{n}} \{s \in S \mid \exists j \in \bar{m}\colon s \circ b_i = f_j\} = \bigcap_{i \in \bar{n}} \bigcup_{j \in \bar{m}} \{s \in S \mid s \circ b_i = f_j\}$ holds. Now notice that $\forall a \in S\colon R_a$ is finite to one and therefore $\forall i \in \bar{n}\colon \forall j \in \bar{m}\colon \{s \in S\colon s \circ b_i = f_j\}$ is finite and sums and intersections of finite set is finite. Therefore $\{s \in S \mid s \circ B \subset F\}$ is finite, contradiction.

□

Jump to proof

Proof that rightcancellative semigroup is leftmoving

Proof.

The semigroup $(S, \circ)$ jest rightcancellative$\Leftrightarrow \forall a, b, s \in S\colon a \neq b \Rightarrow R_s(a) = a \circ s \neq b \circ s = R_s(b) \Leftrightarrow \forall s \in S\colon R_s$ is injection. Therefore for every $s \in S$ preimage of singleton $R_s^{-1}(\{t\})$ is at most singleton and therefore $\forall a \in S\colon R_a$ is finite to one and therefore $(S, \circ)$ is leftmoving.

□

Jump to proof

Proof that rightcancellative semigroup is rightmoving

Proof.

The semigroup $(S, \circ)$ jest rightcancellative$\Leftrightarrow \forall a, b, s \in S\colon a \neq b \Rightarrow R_s(a) = a \circ s \neq b \circ s = R_s(b) \Leftrightarrow \forall s \in S\colon R_s$ is injection. Now suppose that the semigroup $(S, \circ)$ is not rightmoving. Then we find the finite $\{f_1, \ldots, f_m\} = F \subset S$ and the infinite $A \subset S$ such that for every finite $B \subset A$ the set $\{s \in S \mid B \circ s \subset F\} \subset S$ is infinite. We show that this set is empty. For this purpose we choose the arbitrary $b_1, \ldots, b_{2 \cdot m} \in A$ that are different. Then due to rightcancellativity we get that $b_1 \circ s, \ldots, b_{2 \cdot m} \circ s \in S$ are different. □

Jump to proof

The rest of the proof that rightcancellative semigroup is rightmoving

Proof.

If we assume that $\{s \in S \mid \{b_1 \circ s, \dots, b_{2\cdot m} \circ s\} \subset \{f_1, \dots, f_m\}\} \neq \emptyset$ then we get that there is such $s \in S$ that $\{b_1 \circ s, \dots, b_{2\cdot m} \circ s\} \subset \{f_1, \dots, f_m\}$. For this reason we always find such $i, j \in \overline{2\cdot m}$ that $b_i \circ s = f_k = b_j \circ s$, here $k \in \bar{m}$, and subjecting a rightcancellativity comes to us that $b_i = b_j$. This is contradiction with $b_1, \dots, b_{2\cdot m}$ are different. □

Jump to proof

Proof that leftcancellative semigroup is leftmoving

Proof.

The semigroup $(S, \circ)$ jest leftcancellative $\Leftrightarrow \forall a, b, s \in S \colon a \neq b \Rightarrow L_s(a) = s \circ a \neq s \circ b = L_s(b) \Leftrightarrow \forall s \in S \colon L_s$ is injection. Now suppose that the semigroup $(S, \circ)$ is not leftmoving. Then we find the finite $\{f_1, \ldots, f_m\} = F \subset S$ and the infinite $A \subset S$ such that for every finite $B \subset A$ the set $\{s \in S \mid s \circ B \subset F\} \subset S$ is infinite. We show that this set is empty. For this purpose we choose the arbitrary $b_1, \ldots, b_{2 \cdot m} \in A$ that are different. Then due to leftcancellativity we get that $s \circ b_1, \ldots, s \circ b_{2 \cdot m} \in S$ are different. □

Jump to proof

The rest of the proof that leftcancellative semigroup is leftmoving

Proof.

If we assume that $\{s \in S \mid \{s \circ b_1, \dots, s \circ b_{2 \cdot m}\} \subset \{f_1, \dots, f_m\}\} \neq \emptyset$ then we get that there is such $s \in S$ that $\{s \circ b_1, \dots, s \circ b_{2 \cdot m}\} \subset \{f_1, \dots, f_m\}$. For this reason we always find such $i, j \in \overline{2 \cdot m}$ that $s \circ b_i = f_k = s \circ b_j$, here $k \in \bar{m}$, and subjecting a leftcancellativity comes to us that $b_i = b_j$. This is contradiction with $b_1, \dots, b_{2 \cdot m}$ are different. □

Jump to theorem

Proof of uniqueness of Cech Stone compactification

Proof.

Assume that we have two different Cech Stone compactifications (ϕ, W) and (ψ, V). Then for every compact space (Y, τ_Y) and for every continuous function $f \in C(X, Y)$ there is such continuous function $g\colon V \supset \psi(X) \to Y$ that $g \circ \psi = f$ and there is such continuous function $h\colon W \supset \phi(X) \to Y$ that $h \circ \phi = f$. This is happen for every compact space (Y, τ_Y) and every continuous function $f\colon X \to Y$. Here notice that continuous image of compact set is compact. Because W is compact and ϕ is continuous then if we substitute $Y = W, f = \phi$ we get that there is $g\colon V \supset \psi(X) \to \phi(X) \subset W$ that fulfills $g \circ \psi = \phi$. □

Jump to theorem

Proof that $(\beta N, +)$ is a semigroup

Proof.

- This follows from the obvious fact $\forall F \in \beta^g N \colon \exists! n \in N \colon F = e(n)$. Using this fact we get that $\forall A \subset N \colon A \in F + G \Leftrightarrow \exists!(n,m) \in N^2 \colon A \in e(n) + e(m) \Leftrightarrow \{n \in N \mid A - n \in e(m)\} \in e(n) \Leftrightarrow n \in \{n \in N \mid A - n \in e(m)\} \Leftrightarrow A - n \in e(m) \Leftrightarrow m \in A - n \Leftrightarrow m + n \in A \Leftrightarrow A \in e(m+n)$.

□

Jump to proof

The rest of the proof that $(\beta N, +)$ is a semigroup

Proof.

- We show that $\emptyset \notin F + G$. Indeed $\emptyset \in F + G \Leftrightarrow F \ni \{n \in N \mid \emptyset - n \in G\} = \{n \in N \mid \emptyset \in G\} = \emptyset \Leftrightarrow F = 2^X \vee G = 2^X$ a $F, G \in \beta X$. Now we show that $\forall A, B \in 2^N \colon A \in F + G \wedge B \in F + G \Leftrightarrow A \cap B \in F + G$. To proof this notice that $\{n \in N \mid A - n \in G\} \cap \{n \in N \mid B - n \in G\} = \{n \in N \mid A \cap B - n \in G\}$. This equility follows from the equality $\forall m \in N \colon A - m \in G \wedge B - m \in G \Leftrightarrow G \ni (A - m) \cap (B - m) = A \cap B - m$. For every $A, B \subset N$ the following equivalence holds $A \in F + G \wedge B \in F + G \Leftrightarrow F \ni \{n \in N \mid A - n \in G\} \wedge F \ni \{n \in N \mid B - n \in G\} \Leftrightarrow F \ni \{n \in N \mid A - n \in G\} \cap \{n \in N \mid B - n \in G\} = \{n \in N \mid A \cap B - n \in G\} \Leftrightarrow A \cap B \in F + G$.

$\square$

Jump to proof

Proof.

- Now we show the ultrafilter property. Here we use the fact that $(N - A) - n = N - (A - n)$. Notice that $\forall A \in 2^N : A \notin F + G \Leftrightarrow \{n \in N \mid A - n \in G\} \notin F \Leftrightarrow F \ni N - \{n \in N \mid A - n \in G\} = \{n \in N \mid A - n \notin G\} = \{n \in N \mid N - (A - n) \in G\} = \{n \in N \mid (N - A) - n \in G\} \Leftrightarrow N - A \in F + G$.

□

Jump to proof

Proof.

- In order to prove associativity $\forall F, G, H \in \beta N\colon F + (G + H) = (F + G) + H$ at first we show the equality $\{n \in N \mid A - n \in H\} - m = \{n \in N \mid n + m \in \{n \in N \mid A - n \in H\}\} = \{n \in N \mid A - (n + m) \in H\} = \{n \in N \mid A - n - m \in H\}$.
  Now we calculate $\forall A \subset N\colon A \in F + (G + H) \Leftrightarrow \{n \in N \mid A - n \in G + H\} \in F \Leftrightarrow \{n \in N \mid \{m \in N \mid A - n - m \in H\} \in G\} \in F \Leftrightarrow \{n \in N \mid \{m \in N \mid A - m \in H\} - n \in G\} \in F \Leftrightarrow \{m \in N \mid A - m \in H\} \in F + G \Leftrightarrow A \in (F + G) + H$.

□

Jump to proof

The rest of the proof that $(\beta N, +)$ is a semigroup

Proof.

- Assume that we have any set from the Stone base $H \subset \beta N$ that has the form $H = A^\star$ for $A \subset X$. Then we get that $+_G^{-1}(H) = +_G^{-1}(A^\star) = \{F \in \beta N \mid F + G \in A^\star\} = \{F \in \beta N \mid A \in F + G\} = \{F \in \beta N \mid \{n \in N \mid A - n \in G\} \in F\} = \{n \in N \mid A - n \in G\}^\star$ and therefore the preimage of the set from Stone base is in Stone base.

□

Jump to theorem

Proof about idempotent ultrafilters

Proof.

- Denote by $(R, \subset)$ the ordered set of all compact semigroups contained in βN. Then $\beta N \in R$ because $(\beta N, +)$ is semigroup contained in βN and βN is compact. This means that $R \neq \emptyset$. Let $C \subset R$ is arbitrary chain. Notice that every element of C is compact semigroup. Then $\bigcap C$ is the nonempty compact least upper bound of chain C. From Zorn lemma there is minimal element $A \in C \subset R$ and this element A is compact semigroup.

□

▸ Jump to proof

The rest of the proof about idempotent ultrafilters

Proof.

- Notice that for every $F \in A$ the set $A + F = \{X + F \mid X \in A\}$ is compact. To see this recall that $+_F$ is continuous. Therefore $+_F|_A \colon A \to \beta N; X \mapsto +_F|_A(X) = X + F$ is continuous and $+_F|_A(A) = A + F$. As shown A is compact and therefore $A + F$ is compact as the continuous image of compact set. Now we show that $A + F$ is semigroup. Define the operation $\hat{+} \colon (A + F)^2 \to A + F; (X + F, Y + F) \mapsto (X + F)\hat{+}(Y + F) = (X + F + Y) + F$ where $\hat{+} = +|_{A+F}$. Notice that $A + F$ is closed due to $\hat{+}$ because A is subsemigroup of βN and $\forall X, Y, F \in A \colon X + F + Y \in A$. Because $+$ is associative then $\hat{+}$ is associative as restriction of associative operation.

□

▸ Jump to proof

The rest of the proof about idempotent ultrafilters

Proof.

- Now notice that $(A, +_A)$ is semigroup. Therefore $\forall X, F \in A$: $X + F = X +_A F \in A$ and therefore $A + F \subset A$. We showed that A is minimal compact semigroup and $A + F \subset A$ is compact semigroup and so it must be that $A + F = A$. Now recall that for every $F \in A$ the function $+_F : A \to A; G \mapsto +_F(G) = F + G$ is continuous and $B = +_F^{-1}(\{F\}) \subset A$. Every compact space is Hausdorff and every continouos function from compact space to Hausdorff space is proper. This means that preimage of every compact set is compact. The singleton $\{F\}$ is compact and therefore $+_F^{-1}(\{F\})$ is compact because $+_F$ is continuous and proper.

□

Jump to proof

Proof.

- We are going to show that $\forall F \in A\colon F = F + F$. The set $(B, +)$ is semigroup because for every $H_1, H_2 \in B$ holds $H_1 + F = F \wedge H_2 + F = F$ and therefore $H_1 + H_2 + F = H_1 + (H_2 + F) = H_1 + F = F$ that means $H_1 + H_2 \in B$ and $+$ is associative. This means that $B \subset A$ is compact semigroup but A is minimal compact semigroup and therefore $B = A$. This means that $\exists F \in \beta N\colon F \in A \Leftrightarrow F \in B \Leftrightarrow F = F + F$.

□

Jump to proof

The rest of the proof about idempotent ultrafilters

Proof.

- Assume that there is such $F \in \beta N$ that $F \in \beta^g N$ and $F \in \beta^{id} N$. Recall that there is only one such $n \in N$ that $F = e(n)$ and $F = F + F$ and therefore $e(n) = e(n) + e(n) = \{A \subset N \mid \{m \in N \mid A - m \in e(n)\} \in e(n)\} = \{A \subset N \mid n \in \{m \in N \mid n \in A - m\}\} = \{A \subset N \mid n \in \{m \in N \mid n+m \in A\}\} = \{A \subset N \mid 2\cdot n = n+n \in A\} = e(2\cdot n)$ but e is injection and therefore $n = 2 \cdot n$, contradiction with $n \in N$.
- For every $F \in \beta N$ we assume that $F = F + F$. Then for every $A \subset N$ holds $A \in F = F + F \Leftrightarrow \{n \in N \mid A - n \in F\} = \hat{A} \in F$.

□

Jump to proof

The rest of the proof about idempotent ultrafilters

Proof.

- For every $F \in \beta^{id}(N)$ and for every $A \subset N$ holds $A \in F \Leftrightarrow \hat{A} \in F$ and therefore for every $A \in F$ holds $A \in F \wedge \hat{A} \in F$
- For every $F \in \beta^{id}(N)$ and for every $A \subset N$ we get that $A \cap \hat{A} \in F$ and therefore $A \cap \hat{A} \neq \emptyset$.
- For every $A \in F$ holds $A \cap \hat{A} \neq \emptyset$ that means there is such $a \in X$ that $a \in A \cap \hat{A} \Leftrightarrow a \in A \wedge A - a \in F$. Therefore $A \in F \wedge A - a \in F$ and therefore $A \cap (A - a) \in F$.
- Notice that $F \in \beta N^{id} \subset \beta^{w} N$ and we have the obvious fact $\forall F \in \beta^{w} X : \forall a \in X : \forall A \in F : A - \{a\} \in F$. The result is that $A \cap (A - a) - \{a\} \in F$.

□

Jump to theorem

Proof of property of $G-$pair

Proof.

- For each $n \in N$ we get that $K_n = K_{n-1} \cap (K_{n-1} - a_n) - \{a_n\} \subset K_{n-1} \cap (K_{n-1} - a_n) \subset K_{n-1}$ and $K_n = K_{n-1} \cap (K_{n-1} - a_n) - \{a_n\} \subset K_{n-1} \cap (K_{n-1} - a_n) \subset K_{n-1} - a_n$. Hence for every $m \in K_n$ holds $a_n + m \in K_{n-1}$, hence $a_n + K_n \subset K_{n-1}$.

□

Jump to theorem